

CENTRALLY LARGE SUBALGEBRAS AND TRACIAL \mathcal{Z} -ABSORPTION

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ABSTRACT. Let A be a simple infinite dimensional stably finite unital C^* -algebra, and let B be a centrally large subalgebra of A . We prove that if A is tracially \mathcal{Z} -absorbing if and only if B is tracially \mathcal{Z} -absorbing. If A and B are also separable and nuclear, we prove that A is \mathcal{Z} -absorbing if and only if B is \mathcal{Z} -absorbing.

Let \mathcal{Z} be the Jiang-Su algebra. In this paper we prove that if A is a simple infinite dimensional stably finite unital C^* -algebra and $B \subset A$ is a centrally large subalgebra in the sense of [2], then A is tracially \mathcal{Z} -absorbing in the sense of [5] if and only if B is tracially \mathcal{Z} -absorbing. If, in addition, A and B are separable and nuclear, then $\mathcal{Z} \otimes A \cong A$ if and only if $\mathcal{Z} \otimes B \cong B$. (The actual hypotheses are slightly weaker; see Theorem 2.2, Theorem 2.3, Theorem 2.4, and Corollary 2.5.)

Applications will appear elsewhere. The main ones so far are as follows. First, let X be an infinite compact metric space, and let $h: X \rightarrow X$ be a minimal homeomorphism with mean dimension zero. Then $C^*(\mathbb{Z}, X, h)$ is \mathcal{Z} -stable. This is proved by Elliott and Niu in [4], and Theorem 2.3 plays a key role. Second, Theorem 2.3 has been used in [1] to prove \mathcal{Z} -stability of crossed products $C^*(\mathbb{Z}, C(X, D), \alpha)$ when D is simple, unital, and nuclear, the automorphism $\alpha \in \text{Aut}(C(X, D))$ “lies over” a minimal homeomorphism of X (in interesting cases, with large mean dimension), and \mathcal{Z} -stability of the crossed product comes from D rather than from the action of \mathbb{Z} on X . Third, David Kerr has proved ([6]) that if G is a countable infinite amenable group, then there is a free minimal action of G on the Cantor set X with a system of Rokhlin towers which is good enough to construct an AF subalgebra of $C^*(G, X)$ that is centrally large in the sense of Definition 1.1. It follows from Theorem 2.3 that $C^*(G, X)$ is \mathcal{Z} -stable. We also mention Wei Sun’s “generalized higher dimensional noncommutative tori” [15]. Some of these are isomorphic to centrally large subalgebras of crossed products by rotation actions of \mathbb{Z} on $(S^1)^n$. Corollary 2.5 implies that these algebras are \mathcal{Z} -stable, and from this fact one can deduce stable rank one and sometimes deduce real rank zero. See Example 2.6.

Large subalgebras were introduced in [8]. They are an abstraction of an idea whose initial form appeared in [9], and one of the main examples, the orbit breaking subalgebra of a crossed product by a minimal homeomorphism (see Theorem 7.10 of [8]) is a generalization of the construction of [9]. It was shown in [8] that if B

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is stably large in A , then A and B have many properties in common. For example, they have the same tracial states (Theorem 6.2 of [8]), the same quasitraces (Proposition 6.9 of [8]), the same purely positive part of the Cuntz semigroup (Theorem 6.8 of [8]), and the same radius of comparison (Theorem 6.14 of [8]). Centrally large subalgebras, introduced in [2], satisfy an extra condition; the appropriate orbit breaking subalgebras of crossed products by minimal homeomorphisms are centrally large (Theorem 7.10 of [8] and Theorem 4.6 of [2]). If B is centrally large in A and B has stable rank one, then A has stable rank one (Theorem 6.3 of [2]), and if in addition B has real rank zero, then the same is true of A (Theorem 6.3 of [2]). This paper extends the results above by proving that tracial \mathcal{Z} -stability and, in the separable nuclear case, \mathcal{Z} -stability, pass between a stably centrally large subalgebra and the containing algebra. We do not know whether it suffices to consider just a large subalgebra.

In Section 1 we recall the definitions and prove several technical results related to approximate commutation relations and order zero maps. Corollary 1.6 gives a very explicit characterization of which linear maps from M_n are completely positive contractive with order zero. In Section 2 we recall the definition of tracial \mathcal{Z} -absorption and prove the main results.

We use Section 1 of [8] as a general reference for facts about Cuntz comparison, although the results we use are not new there. Also, we will repeatedly use without comment the fact (Proposition 5.2 of [8]) that if B is a large subalgebra of a simple unital C^* -algebra A , then B is simple.

1. CENTRALLY LARGE SUBALGEBRAS AND APPROXIMATE COMMUTATION

We recall the definitions of large and centrally large subalgebras (Definition 4.1 of [8] and Definition 3.2 of [2]).

Definition 1.1. Let A be a simple unital infinite dimensional C^* -algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $\varepsilon > 0$, every $m \in \mathbb{Z}_{>0}$, all $a_1, a_2, \dots, a_m \in A$, every $y \in B_+ \setminus \{0\}$, and every $x \in A_+ \setminus \{0\}$ with $\|x\| = 1$, there exist $c_1, c_2, \dots, c_m \in A$ and $g \in B$ such that:

- (1) $0 \leq g \leq 1$.
- (2) $\|c_j - a_j\| < \varepsilon$ for $j = 1, 2, \dots, m$.
- (3) $(1 - g)c_j \in B$ for $j = 1, 2, \dots, m$.
- (4) $g \precsim_B y$ and $g \precsim_A x$.
- (5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

We say that B is *centrally large* in A if we can in addition arrange that:

- (6) $\|ga_j - a_jg\| < \varepsilon$ for $j = 1, 2, \dots, m$.

Lemma 1.2 below is a strengthening of Definition 1.1 in the following ways. The element g can be replaced by a tower of elements, as shown in Lemma 6.1 of [2]. Additionally, by proceeding as in the proof of Lemma 4.7 of [8] at the appropriate step, it can be arranged that the elements c_j which are chosen satisfy $\|c_j\| \leq \|a_j\|$. We omit the details of the proof; see [2] and [8].

Lemma 1.2. Let A be an infinite dimensional simple separable unital C^* -algebra, and let $B \subset A$ be a centrally large subalgebra. Then for all $m, N \in \mathbb{Z}_{>0}$, all $a_1, a_2, \dots, a_m \in A$, every $\varepsilon > 0$, every $x \in A_+$ with $\|x\| = 1$, and every $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \dots, c_m \in A$ for $j = 1, 2, \dots, m$ and $g_0, g_1, \dots, g_N \in B$ such that:

- (1) $0 \leq g_n \leq 1$ for $n = 0, 1, \dots, N$ and $g_{n-1}g_n = g_n$ for $n = 1, 2, \dots, N$.
- (2) For $j = 1, 2, \dots, m$ we have $\|c_j - a_j\| < \varepsilon$.
- (3) For $j = 1, 2, \dots, m$ and $n = 0, 1, \dots, N$, we have $(1 - g_n)c_j \in B$.
- (4) For $n = 0, 1, \dots, N$, we have $g_n \precsim_B y$ and $g_n \precsim_A x$.
- (5) For $n = 0, 1, \dots, N$, we have $\|(1 - g_n)x(1 - g_n)\| > 1 - \varepsilon$.
- (6) For $j = 1, 2, \dots, m$ and $n = 0, 1, \dots, N$, we have $\|g_n a_j - a_j g_n\| < \varepsilon$.
- (7) For $j = 1, 2, \dots, m$ we have $\|c_j\| \leq \|a_j\|$.

As was shown in the proof of Lemma 6.1 of [2], it is enough to take $n = 0$ in (4) and (5) and $n = N$ in (3).

In the definition of tracial \mathcal{Z} -absorption, one needs to control the norms of certain commutators in A in terms of norms of commutators in B . The following lemma contains the basic estimate. We will combine it with the choice $N = 1$ in Lemma 1.2.

Lemma 1.3. For every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. Let A be a C^* -algebra, and let $z, z_0, c, g_0, g_1 \in A$ satisfy:

- (1) $0 \leq g_1 \leq g_0 \leq 1$.
- (2) $g_0 g_1 = g_1$.
- (3) $\|z\| \leq 1$, $\|z_0\| \leq 1$, and $\|c\| \leq 1$.
- (4) $\|[c, g_0]\| < \delta$ and $\|[c, g_1]\| < \delta$.
- (5) $\|[z_0, (1 - g_1)c]\| < \delta$.
- (6) $\|[z_0, g_0]\| < \delta$.
- (7) $\|z - (1 - g_0)^{1/2} z_0 (1 - g_0)^{1/2}\| < \delta$.

Then $\|[z, c]\| < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. Apply Lemma 2.5 of [2] with $\frac{\varepsilon}{11}$ in place of ε and with the function $f(\lambda) = \lambda^{1/2}$, obtaining $\delta_0 > 0$ such that whenever D is a C^* -algebra and $a, b \in D$ satisfy the relations $\|[a, b]\| < \delta_0$, $0 \leq a \leq 1$, and $\|b\| \leq 1$, then $\|[a^{1/2}, b]\| < \frac{\varepsilon}{11}$. Set $\delta = \min(\frac{\varepsilon}{11}, \delta_0)$. This is the number whose existence is asserted in the lemma.

Now let A be a C^* -algebra, and let $z, z_0, c, g_0, g_1 \in A$ satisfy (1)–(7).

We begin by estimating

$$\|[(1 - g_0)^{1/2} z_0 (1 - g_0)^{1/2}, (1 - g_1)c]\|.$$

Using (6) and the choice of δ_0 , we get

$$\|z_0 (1 - g_0)^{1/2} - (1 - g_0)^{1/2} z_0\| < \frac{\varepsilon}{11}.$$

Since all the terms have norm at most 1 (by (1) and (3)), we can use this relation twice at the second step, use (5) and (4) at the third step, and use $g_0 g_1 = g_1 g_0$ (from (1) and (2)), to get

$$\begin{aligned} & \|[(1 - g_0)^{1/2} z_0 (1 - g_0)^{1/2}, (1 - g_1)c]\| \\ &= \|(1 - g_0)^{1/2} z_0 (1 - g_0)^{1/2} (1 - g_1)c - (1 - g_1)c (1 - g_0)^{1/2} z_0 (1 - g_0)^{1/2}\| \\ &< \frac{2\varepsilon}{11} + \|(1 - g_0)z_0(1 - g_1)c - (1 - g_1)c(1 - g_0)z_0\| \\ &< \frac{2\varepsilon}{11} + \delta + \delta + \|(1 - g_0)(1 - g_1)cz_0 - (1 - g_1)(1 - g_0)cz_0\| \\ &= \frac{2\varepsilon}{11} + \delta + \delta \leq \frac{4\varepsilon}{11}. \end{aligned}$$

Since all the terms have norm at most 1, it now follows from (7) that

$$(1.1) \quad \|[z, (1 - g_1)c]\| < \frac{4\varepsilon}{11} + 2\|z - (1 - g_0)^{1/2}z_0(1 - g_0)^{1/2}\| < \frac{4\varepsilon}{11} + 2\delta \leq \frac{6\varepsilon}{11}.$$

Next, we estimate $\|z - (1 - g_1)z\|$ and $\|z - z(1 - g_1)\|$. The relation

$$(1 - g_1)(1 - g_0) = (1 - g_0)(1 - g_1) = 1 - g_0$$

(from (2)) implies that

$$(1 - g_1)(1 - g_0)^{1/2}z_0(1 - g_0)^{1/2} = (1 - g_0)^{1/2}z_0(1 - g_0)^{1/2}.$$

and

$$(1 - g_0)^{1/2}z_0(1 - g_0)^{1/2}(1 - g_1) = (1 - g_0)^{1/2}z_0(1 - g_0)^{1/2}.$$

Therefore, using (7) and $\|1 - g_1\| \leq 1$ at the second step,

$$(1.2) \quad \|z - (1 - g_1)z\| \leq (1 + \|1 - g_1\|)\|z - (1 - g_0)^{1/2}z_0(1 - g_0)^{1/2}\| < 2\delta.$$

Similarly

$$(1.3) \quad \|z - z(1 - g_1)\| < 2\delta.$$

At the second step in the following calculation, we use (1.3) and $\|c\| \leq 1$ on the first term, (1.1) on the second term, (4) and $\|z\| \leq 1$ on the third term, and (1.2) on the fourth term:

$$\begin{aligned} \|[z, c]\| &\leq \|z - z(1 - g_1)\| \|c\| + \|z(1 - g_1)c - (1 - g_1)cz\| \\ &\quad + \|(1 - g_1)c - c(1 - g_1)\| \|z\| + \|c\| \|(1 - g_1)z - z\| \\ &< 2\delta + \frac{6\varepsilon}{11} + \delta + 2\delta \leq \varepsilon. \end{aligned}$$

This completes the proof. \square

Lemma 1.7 below is closely related to Lemma 1.2.5 of [17]. The proof given in [17] is very sketchy. In particular, the elements in the range of φ are not in the linear span of the images of the generators of CM_n used there, only in the C^* -algebra they generate. We address this issue by using a different presentation of CM_n . The following notation is convenient.

Notation 1.4. For $n \in \mathbb{Z}_{>0}$, let $(e_{j,k})_{j,k=1,2,\dots,n}$ be the standard system of matrix units for M_n . We take the cone CM_n over M_n to be

$$CM_n = C_0((0, 1]) \otimes M_n = \{f \in C([0, 1], M_n) : f(0) = 0\}.$$

We let $t \in C_0((0, 1])$ be the function $t(\lambda) = \lambda$ for $\lambda \in (0, 1]$. For $j, k = 1, 2, \dots, n$ we define $f_{j,k} \in CM_n$ by $f_{j,k} = t \otimes e_{j,k}$.

If A is a C^* -algebra, we denote its unitization by A^+ , adding a new identity even if A is already unital.

Lemma 1.5. Let $n \in \mathbb{Z}_{>0}$. Let C be the universal C^* -algebra on generators $x_{j,k}$ for $j, k = 1, 2, \dots, n$, subject to the following relations:

- (1) $x_{j,k}x_{k,m} = x_{j,l}x_{l,m}$ for $j, k, l, m = 1, 2, \dots, n$.
- (2) $x_{j,j}x_{k,k} = 0$ for $j, k = 1, 2, \dots, n$ with $j \neq k$.
- (3) $x_{j,k} = x_{k,j}^*$ for $j, k = 1, 2, \dots, n$.
- (4) $\|x_{j,j}\| \leq 1$ for $j = 1, 2, \dots, n$.
- (5) $\|1 - x_{j,j}\| \leq 1$ for $j = 1, 2, \dots, n$.

Then there is an isomorphism $\rho: C \rightarrow (CM_n)^+$ such that $\rho(x_{j,k}) = f_{j,k}$ for $j, k = 1, 2, \dots, n$.

Proof. We claim that the relations in the statement imply the following additional relations:

- (6) $x_{j,j} \geq 0$ for $j = 1, 2, \dots, n$.
- (7) $\|x_{j,k}\| \leq 1$ for $j, k = 1, 2, \dots, n$.
- (8) $x_{j,k} = \lim_{n \rightarrow \infty} x_{j,j}^{1/n} x_{j,k} = \lim_{n \rightarrow \infty} x_{j,k} x_{k,k}^{1/n}$ for $j, k = 1, 2, \dots, n$.
- (9) $x_{j,k} x_{l,m} = 0$ for $j, k, l, m = 1, 2, \dots, n$ with $k \neq l$.

Relation (6) follows from (3) (which implies that $x_{j,j}$ is selfadjoint), (4), and (5). To prove (7), use (3), (1), and (4) to see that $\|x_{j,k}^* x_{j,k}\| = \|x_{k,k}^2\| \leq 1$. For the first part of (8), use (1), (3), and (6), to get $x_{j,j} = (x_{j,k} x_{j,k}^*)^{1/2}$; now use the general fact $\lim_{n \rightarrow \infty} (aa^*)^{1/n} a = a$ for any element a of any C*-algebra. The second part of (8) follows by taking adjoints and using (3). To prove (9), we now use (8) at the first step and (2) at the second step to get

$$x_{j,k} x_{l,m} = \lim_{n \rightarrow \infty} x_{j,k} x_{k,k}^{1/n} x_{l,l}^{1/n} x_{l,m} = \lim_{n \rightarrow \infty} x_{j,k} \cdot 0 \cdot x_{l,m} = 0.$$

This completes the proof of the claim.

In the rest of the proof, we follow Notation 1.4.

It is immediate that there is a unital homomorphism $\rho: C \rightarrow (CM_n)^+$ such that $\rho(x_{j,k}) = f_{j,k}$ for $j, k = 1, 2, \dots, n$. If $n = 1$ then (1) and (2) are vacuous, (3) says that $x_{1,1}$ is selfadjoint, (4) says that $\|x_{1,1}\| \leq 1$, and (5) then just says that $x_{1,1}$ is positive. The conclusion is now easy. So assume $n \geq 2$.

Let D be the universal C*-algebra on generators y_j for $j = 2, 3, \dots, n$, subject to the relations:

- (10) $y_j^* y_j = y_2^* y_2$ for $j = 3, 4, \dots, n$.
- (11) $y_j y_k = 0$ for $j, k = 2, 3, \dots, n$.
- (12) $y_j^* y_k = 0$ for $j, k = 2, 3, \dots, n$ with $j \neq k$.
- (13) $\|y_j\| \leq 1$ for $j = 2, 3, \dots, n$.

Recall from Proposition 3.3.1 of [7] that there is an isomorphism $\sigma: CM_n \rightarrow D$ such that $\sigma(f_{j,1}) = y_j$ for $j = 2, 3, \dots, n$. Examining the relations (1), (3), (7), and (9), one sees that there is a unital homomorphism $\mu: (CM_n)^+ \rightarrow C$ such that $\mu(f_{j,1}) = x_{j,1}$ for $j = 2, 3, \dots, n$. Clearly $\rho \circ \mu$ is the identity map on $(CM_n)^+$. The lemma will thus be proved if we show that μ is surjective.

Define $B = \mu((CM_n)^+) \subset C$. Obviously $1 \in B$, and $x_{j,1} \in B$ for $j = 2, 3, \dots, n$. It follows from (3) that $x_{1,j} \in B$ for $j = 2, 3, \dots, n$. For $j = 1, 2, \dots, n$, recall that $x_{j,j} \geq 0$ by (6). If $j \neq 1$, the relation (1) implies that $x_{j,j}^2 = x_{j,1} x_{1,j}$. Thus $x_{j,j}^2 \in B$, whence $x_{j,j} = (x_{j,j}^2)^{1/2} \in B$. For $j = 1$ we use (1) to get instead $x_{1,1}^2 = x_{1,2} x_{2,1} \in B$, whence $x_{1,1} \in B$ as before.

Now let $j, k \in \{1, 2, \dots, n\}$ be arbitrary. For $r \in \mathbb{Z}_{>0}$ we use (1) to get

$$(1.4) \quad x_{j,j}^r x_{j,k} = x_{j,j}^{r-1} x_{j,1} x_{1,k} \in B.$$

For $\alpha \in (0, \infty)$ we can approximate $\lambda \mapsto \lambda^\alpha$ on $[0, 1]$ by polynomials with no constant term, and therefore deduce from (1.4) that $x_{j,j}^\alpha x_{j,k} \in B$. Now $x_{j,k} \in B$ follows from (8). We have shown that μ is surjective, and proved the lemma. \square

We can now give a very explicit characterization of completely positive contractive order zero maps from M_n . For completeness, we include the already known characterization in terms of homomorphisms from CM_n .

Corollary 1.6. Let A be a C^* -algebra, let $n \in \mathbb{Z}_{>0}$, and let $\varphi: M_n \rightarrow A$ be a linear map. Then the following are equivalent (using Notation 1.4 in (2), (3), and (4)):

- (1) φ is completely positive contractive and has order zero.
- (2) There is a homomorphism $\pi: CM_n \rightarrow A$ such that $\varphi(z) = \pi(t \otimes z)$ for all $z \in M_n$.
- (3) The elements $x_{j,k} = \varphi(e_{j,k})$ for $j, k = 1, 2, \dots, n$ satisfy the relations (1)–(5) of Lemma 1.5 as elements of the C^* -algebra A^+ .
- (4) The elements $x_{j,k} = \varphi(e_{j,k})$ for $j, k = 1, 2, \dots, n$ satisfy the relations (1)–(4) of Lemma 1.5, together with the relation $x_{j,j} \geq 0$ for $j = 1, 2, \dots, n$, as elements of the C^* -algebra A .

Proof. The equivalence of (1) and (2) is Corollary 4.1 of [18] (or the proposition in 1.2.2 of [17]). That (2) implies (4) is clear, as is the implication from (4) to (3). So assume (3). Lemma 1.5 provides a unital homomorphism $\sigma: (CM_n)^+ \rightarrow A^+$ such that $\sigma(t \otimes e_{j,k}) = \varphi(e_{j,k})$ for $j, k = 1, 2, \dots, n$. Since $\varphi(e_{j,k}) \in A$ for $j, k = 1, 2, \dots, n$, it follows that $\pi = \sigma|_{CM_n}$ is a homomorphism from CM_n to A such that $\pi(t \otimes e_{j,k}) = \varphi(e_{j,k})$ for $j, k = 1, 2, \dots, n$. Condition (2) is now immediate. \square

Lemma 1.7. For every $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. Let A be a C^* -algebra, let $B \subset A$ be a subalgebra, let $n \in \mathbb{Z}_{>0}$, let $\varphi_0: M_n \rightarrow A$ be a completely positive contractive order zero map, and let $x \in B$ satisfy:

- (1) $0 \leq x \leq 1$.
- (2) With $e_{j,k}$ as in Notation 1.4, we have $\|[x, \varphi_0(e_{j,k})]\| < \delta$ for $j, k = 1, 2, \dots, n$.
- (3) $\text{dist}(\varphi_0(e_{j,k})x, B) < \delta$ for $j, k = 1, 2, \dots, n$.

Then there is a completely positive contractive order zero map $\varphi: M_n \rightarrow B$ such that for all $z \in M_n$ with $\|z\| \leq 1$, we have $\|\varphi_0(z)x - \varphi(z)\| < \varepsilon$.

Proof. We first consider the case $n = 1$ separately, since one step of the argument for $n \geq 2$ doesn't work in this case.

Define a continuous function $f: \mathbb{R} \rightarrow [0, 1]$ by

$$f(\lambda) = \begin{cases} 0 & \lambda \leq 0 \\ \lambda & 0 \leq \lambda \leq 1 \\ 1 & 1 \leq \lambda. \end{cases}$$

By a polynomial approximation argument, there is $\delta_0 > 0$ such that $\delta_0 \leq \min(1, \varepsilon)$ and whenever A is a C^* -algebra and $c, d \in A_{\text{sa}}$ satisfy

$$\|c\| \leq 2, \quad \|d\| \leq 2, \quad \text{and} \quad \|c - d\| < \delta_0,$$

then $\|f(c) - f(d)\| < \frac{\varepsilon}{2}$. Use Lemma 2.5 of [2] to choose $\delta > 0$ such that $\delta \leq \frac{\delta_0}{2}$ and whenever A is a C^* -algebra and $a, x \in A$ satisfy

$$\|a\| \leq 1, \quad \|x\| \leq 1, \quad x \geq 0, \quad \text{and} \quad \|[x, a]\| < \delta,$$

then $\|[x^{1/2}, a]\| < \frac{\delta_0}{2}$.

Now let $A, B, \varphi_0: \mathbb{C} \rightarrow A$, and x be as in the hypotheses. Then

$$(1.5) \quad \|\varphi_0(1)x - x^{1/2}\varphi_0(1)x^{1/2}\| \leq \|[x^{1/2}, \varphi_0(1)]\| \|x^{1/2}\| < \frac{\delta_0}{2}$$

and

$$\text{dist}(\varphi_0(1)x, B) < \frac{\delta_0}{2}.$$

These inequalities allow us to choose $d_0 \in B$ such that $\|d_0 - x^{1/2}\varphi_0(1)x^{1/2}\| < \delta_0$. Then $d = \frac{1}{2}(d_0 + d_0^*)$ also satisfies $\|d - x^{1/2}\varphi_0(1)x^{1/2}\| < \delta_0$. In particular, $\|d\| \leq 2$. Since

$$f(x^{1/2}\varphi_0(1)x^{1/2}) = x^{1/2}\varphi_0(1)x^{1/2},$$

the choice of δ_0 implies that $\|f(d) - x^{1/2}\varphi_0(1)x^{1/2}\| < \frac{\varepsilon}{2}$. Combining this estimate with (1.5) and $\delta_0 \leq \varepsilon$ gives $\|f(d) - \varphi_0(1)x\| < \varepsilon$. Now we can define $\varphi: \mathbb{C} \rightarrow B$ by $\varphi(\lambda) = \lambda f(d)$ for $\lambda \in \mathbb{C}$.

Now assume $n \geq 2$. For $\delta_0 > 0$ consider the following relations on elements $y_{j,k}$ in a C^* -algebra, for $j, k = 1, 2, \dots, n$:

- (1) $\|y_{j,k}y_{k,m} - y_{j,l}y_{l,m}\| < \delta_0$ for $j, k, l, m = 1, 2, \dots, n$.
- (2) $\|y_{j,j}y_{k,k}\| < \delta_0$ for $j, k = 1, 2, \dots, n$ with $j \neq k$.
- (3) $\|y_{j,k} - y_{k,j}^*\| < \delta_0$ for $j, k = 1, 2, \dots, n$.
- (4) $\|y_{j,j}\| < 1 + \delta_0$ for $j = 1, 2, \dots, n$.
- (5) $\|1 - y_{j,j}\| < 1 + \delta_0$ for $j = 1, 2, \dots, n$.

Since the cone CM_n is projective (see Theorem 10.2.1 of [7]), it is semiprojective, so its unitization $(CM_n)^+$ is semiprojective (by Theorem 14.1.7 of [7]). It therefore follows from Theorem 14.1.4 of [7] that the relations in Lemma 1.5 are stable in the sense of Definition 14.1.1 of [7]. Thus, there is $\delta_0 > 0$ such that, whenever D is a unital C^* -algebra and $y_{j,k}$, for $j, k = 1, 2, \dots, n$, are elements of D satisfying (1)–(5) above, then there is a unital homomorphism $\sigma: (CM_n)^+ \rightarrow D$ such that, with $f_{j,k}$ as defined in Notation 1.4, we have

$$(1.6) \quad \|\sigma(f_{j,k}) - y_{j,k}\| < \frac{\varepsilon}{2n^2}$$

for $j, k = 1, 2, \dots, n$. Use Lemma 2.5 of [2] to choose $\delta > 0$ such that

$$(1.7) \quad \delta \leq \min\left(1, \frac{\delta_0}{8}, \frac{\varepsilon}{2n^2}\right)$$

and whenever A is a C^* -algebra and $a, x \in A$ satisfy

$$\|a\| \leq 1, \quad \|x\| \leq 1, \quad x \geq 0, \quad \text{and} \quad \|[x, a]\| < \delta,$$

then $\|[x^{1/2}, a]\| < \frac{\delta_0}{2}$.

Now let $A, B, \varphi_0: M_n \rightarrow A$, and x be as in the hypotheses. For $j, k = 1, 2, \dots, n$ choose $y_{j,k} \in B$ such that

$$(1.8) \quad \|y_{j,k} - \varphi_0(e_{j,k})x\| < \delta.$$

Then $\|y_{j,k}\| < 1 + \delta$. We claim that the relations (1)–(5) above are satisfied in B^+ . First, by Corollary 1.6(3), the elements $x_{j,k} = \varphi_0(e_{j,k})$ satisfy the relations (1)–(5) of Lemma 1.5 as elements of the C^* -algebra A^+ . We now verify (1). We have, using (1.8) and

$$\|\varphi_0(e_{j,k})x\| \leq 1, \quad \|y_{j,k}\| < 1 + \delta, \quad \text{and} \quad \|y_{j,l}\| < 1 + \delta$$

at the first step,

$$\begin{aligned} & \|y_{j,k}y_{k,m} - y_{j,l}y_{l,m}\| \\ & < 2\delta(1 + \delta) + 2\delta + \|\varphi_0(e_{j,k})x\varphi_0(e_{k,m})x - \varphi_0(e_{j,l})x\varphi_0(e_{l,m})x\| \\ & \leq 2\delta(2 + \delta) + \|[\varphi_0(e_{k,m}), x]\| + \|[\varphi_0(e_{l,m}), x]\| \\ & \quad + \|\varphi_0(e_{j,k})\varphi_0(e_{k,m})x^2 - \varphi_0(e_{j,l})\varphi_0(e_{l,m})x^2\|. \end{aligned}$$

The last term in the last expression is zero by condition (1) in Lemma 1.5, and, using hypothesis (2) at the first step and (1.7) at the second and third steps,

$$2\delta(2 + \delta) + \|[\varphi_0(e_{k,m}), x]\| + \|[\varphi_0(e_{l,m}), x]\| < 2\delta(2 + \delta) + 2\delta \leq 8\delta \leq \delta_0.$$

Thus (1) holds. Similarly, using, in order, (2), (3), and (4) in Lemma 1.5, for $j, k = 1, 2, \dots, n$ with $j \neq k$ we have

$$\begin{aligned} & \|y_{j,j}y_{k,k}\| < \delta(1 + \delta) + \delta + \|\varphi_0(e_{j,j})x\varphi_0(e_{k,k})x\| \\ & < \delta(1 + \delta) + \delta + \delta + \|\varphi_0(e_{j,j})\varphi_0(e_{k,k})x^2\| = \delta(3 + \delta) \leq 4\delta < \delta_0, \end{aligned}$$

for $j, k = 1, 2, \dots, n$ we have

$$\|y_{j,k} - y_{k,j}^*\| < 2\delta + \|\varphi_0(e_{j,k})x - x\varphi_0(e_{k,j})^*\| < 3\delta + \|\varphi_0(e_{j,k})x - \varphi_0(e_{k,j})^*x\| < \delta_0,$$

and for $j = 1, 2, \dots, n$ we have

$$\|y_{j,j}\| < \delta + \|\varphi_0(e_{j,j})x\| \leq 1 + \delta < 1 + \delta_0.$$

Finally, for $j = 1, 2, \dots, n$, the choice of δ and hypothesis (2) imply that $\|[x^{1/2}, \varphi_0(x_{j,j})]\| < \frac{\delta_0}{2}$, so

$$\begin{aligned} & \|y_{j,j} - x^{1/2}\varphi_0(x_{j,j})x^{1/2}\| \leq \|y_{j,j} - \varphi_0(x_{j,j})x\| + \|[x^{1/2}, \varphi_0(x_{j,j})]\| \|x^{1/2}\| \\ & < \delta + \frac{\delta_0}{2} \leq \delta_0. \end{aligned}$$

Since $0 \leq x^{1/2}\varphi_0(x_{j,j})x^{1/2} \leq 1$, we have

$$\|y_{j,j}\| \leq 1 + \delta_0 \quad \text{and} \quad \|1 - x^{1/2}\varphi_0(x_{j,j})x^{1/2}\| \leq 1.$$

Therefore $\|1 - y_{j,j}\| < 1 + \delta_0$. This completes the verification of (1)–(5).

By the choice of δ_0 , there is a unital homomorphism $\sigma: (CM_n)^+ \rightarrow B^+$ such that (1.6) holds for $j, k = 1, 2, \dots, n$. Since $n \geq 2$, there are no nonzero homomorphisms from CM_n to \mathbb{C} . It follows that the formula $\varphi(z) = \sigma(t \otimes z)$, for $z \in M_n$, defines a completely positive contractive order zero map from M_n to B . For $j, k = 1, 2, \dots, n$, we have, using (1.6) and (1.8) at the second step, and (1.7) at the third step,

$$\|\varphi(e_{j,k}) - \varphi_0(e_{j,k})x\| \leq \|\varphi(e_{j,k}) - y_{j,k}\| + \|y_{j,k} - \varphi_0(e_{j,k})x\| < \frac{\varepsilon}{2n^2} + \delta \leq \frac{\varepsilon}{n^2}.$$

Now let $z \in M_n$ satisfy $\|z\| \leq 1$. Choose $\lambda_{j,k} \in \mathbb{C}$ for $j, k = 1, 2, \dots, n$ such that $z = \sum_{j,k=1}^n \lambda_{j,k}e_{j,k}$. One easily sees that $|\lambda_{j,k}| \leq 1$ for $j, k = 1, 2, \dots, n$. So

$$\|\varphi(z) - \varphi_0(z)x\| \leq \sum_{j,k=1}^n |\lambda_{j,k}| \|\varphi(e_{j,k}) - \varphi_0(e_{j,k})x\| < n^2 \left(\frac{\varepsilon}{n^2} \right) = \varepsilon.$$

This completes the proof. \square

For unital C^* -algebras, we strengthen Lemma 1.7 by adding a condition to the conclusion, as follows.

Lemma 1.8. For every $\varepsilon > 0$ and $n \in \mathbb{Z}_{>0}$, there is $\delta > 0$ such that the following holds. Whenever $A, B, \varphi_0: M_n \rightarrow A$, and $x \in B$ satisfy the conditions in Lemma 1.7, and in addition A is unital and B contains the identity of A , there exists a completely positive contractive order zero map $\varphi: M_n \rightarrow B$ such that:

- (1) $\|\varphi(z) - \varphi_0(z)x\| < \varepsilon$ for all $z \in M_n$ with $\|z\| \leq 1$.
- (2) $1 - \varphi(1) \precsim_A (1 - x) \oplus [1 - \varphi_0(1)]$.

Proof. Set $\varepsilon_0 = \min(\frac{1}{3}, \frac{\varepsilon}{3})$. Use Lemma 2.5 of [2] to choose $\delta_0 > 0$ such that whenever A is a C^* -algebra and $a, x \in A$ satisfy

$$\|a\| \leq 1, \quad \|x\| \leq 1, \quad x \geq 0, \quad \text{and} \quad \|[x, a]\| < \delta_0,$$

then $\|[x^{1/2}, a]\| < \varepsilon_0$. Apply Lemma 1.7 with $\min(\varepsilon_0, \delta_0)$ in place of ε , getting $\delta_1 > 0$. Set $\delta = \min(\delta_0, \delta_1)$.

Now let A be a unital C^* -algebra, and let $B \subset A$, $\varphi_0: M_n \rightarrow A$, and $x \in B$ be as in the hypotheses. The choice of δ_1 using Lemma 1.7 gives us a completely positive contractive order zero map $\varphi_1: M_n \rightarrow B$ such that

$$(1.9) \quad \|\varphi_1(z) - \varphi_0(z)x\| < \min(\varepsilon_0, \delta_0)$$

for all $z \in M_n$ with $\|z\| \leq 1$. The conditions on φ_0 imply that $\|[\varphi_0(1), x]\| < \delta_0$, so by the choice of δ_0 we have $\|[\varphi_0(1), x^{1/2}]\| < \varepsilon_0$. Combining this estimate with the case $z = 1$ of (1.9), we get

$$(1.10) \quad \|\varphi_1(1) - x^{1/2}\varphi_0(1)x^{1/2}\| < 2\varepsilon_0.$$

Define a continuous function $f: [0, 1] \rightarrow [0, 1]$ by

$$f(\lambda) = \begin{cases} (1 - 2\varepsilon_0)^{-1}\lambda & 0 \leq \lambda \leq 1 - 2\varepsilon_0 \\ 1 & 1 - 2\varepsilon_0 \leq \lambda \leq 1. \end{cases}$$

Following the functional calculus for completely positive order zero maps in Corollary 4.2 of [18], define a completely positive contractive order zero map $\varphi: M_n \rightarrow B$ by $\varphi = f(\varphi_1)$.

We verify part (1) of the conclusion. Let $C \subset B$ be the C^* -algebra generated by $\varphi_1(M_n)$. Theorem 3.3 of [18] provides a homomorphism $\pi: M_n \rightarrow M(C)$ (the multiplier algebra of C) whose range commutes with $\varphi_1(1)$ and such that

$$(1.11) \quad \varphi_1(z) = \pi(z)\varphi_1(1)$$

for all $z \in M_n$. Therefore $\pi(1)\varphi_1(1) = \varphi_1(1)$. For any continuous function $g: [0, \infty) \rightarrow \mathbb{C}$ with $g(0) = 0$, approximation by polynomials with no constant term gives

$$(1.12) \quad g(\varphi_1(1)) = \pi(1)g(\varphi_1(1)).$$

By definition (see Corollary 4.2 of [18]), we have

$$(1.13) \quad \varphi(z) = \pi(z)f(\varphi_1(1))$$

for all $z \in M_n$. In particular, $\varphi(1) = \pi(1)f(\varphi_1(1))$, so (1.12) implies that $\varphi(1) = f(\varphi_1(1))$. Since $|f(\lambda) - \lambda| \leq 2\varepsilon_0$ for all $\lambda \in [0, 1]$, we have $\|\varphi_1(1) - f(\varphi_1(1))\| \leq 2\varepsilon_0$. Combining this estimate with (1.11) and (1.13) gives $\|\varphi(z) - \varphi_1(z)\| \leq 2\varepsilon_0\|z\|$ for all $z \in M_n$. Using (1.9), we now get $\|\varphi(z) - \varphi_0(z)x\| < 3\varepsilon_0 \leq \varepsilon$ for all $z \in M_n$ with $\|z\| \leq 1$, as desired.

It remains to verify part (2) of the conclusion. For $\lambda \in [0, 1]$, we have

$$1 - f(\lambda) = (1 - 2\varepsilon_0)^{-1} \max(0, 1 - \lambda - 2\varepsilon_0).$$

Since $\varphi(1) = f(\varphi_1(1))$, it follows that

$$(1.14) \quad 1 - \varphi(1) \sim_A (1 - \varphi_1(1) - 2\varepsilon_0)_+.$$

By (1.10), we have

$$\|[1 - \varphi_1(1)] - [1 - x^{1/2}\varphi_0(1)x^{1/2}]\| < 2\varepsilon_0,$$

whence

$$(1.15) \quad (1 - \varphi(1) - 2\varepsilon_0)_+ \precsim_A 1 - x^{1/2}\varphi_0(1)x^{1/2}.$$

Moreover, using Lemma 1.4(4) of [8] at the third step, we get

$$\begin{aligned} 1 - x^{1/2}\varphi_0(1)x^{1/2} &= 1 - x + x^{1/2}[1 - \varphi_0(1)]x^{1/2} \\ &\precsim_A (1 - x) \oplus x^{1/2}[1 - \varphi_0(1)]x^{1/2} \\ &\sim_A (1 - x) \oplus [1 - \varphi_0(1)]^{1/2}x[1 - \varphi_0(1)]^{1/2} \\ &\leq (1 - x) \oplus [1 - \varphi_0(1)]. \end{aligned}$$

Combining this result with (1.14) and (1.15) gives

$$1 - \varphi(1) \precsim_A (1 - x) \oplus [1 - \varphi_0(1)],$$

as desired. \square

2. TRACIAL \mathcal{Z} -ABSORPTION

In this section, we prove our main results.

The following definition first appeared in [5].

Definition 2.1 (Definition 2.1 of [5]). Let A be a unital C^* -algebra. We say that A is *tracially \mathcal{Z} -absorbing* if $A \not\cong \mathbb{C}$ and for any $\varepsilon > 0$, any finite set $F \subset A$, any $n \in \mathbb{Z}_{>0}$, and any $x \in A_+ \setminus \{0\}$, there is a completely positive contractive order zero map $\varphi: M_n \rightarrow A$ such that:

- (1) $1 - \varphi(1) \precsim_A x$.
- (2) For any $y \in M_n$ with $\|y\| = 1$ and any $a \in F$, we have $\|\varphi(y)a - a\varphi(y)\| < \varepsilon$.

Theorem 2.2. Let A be a simple infinite dimensional unital C^* -algebra, and let B be a centrally large subalgebra of A . If B is tracially \mathcal{Z} -absorbing, then A is tracially \mathcal{Z} -absorbing.

We don't need to assume that A is finite. Theorem 3.3 of [5] shows that if B is tracially \mathcal{Z} -absorbing, then B has strict comparison of positive elements, from which it follows that B either has a normalized quasitrace or is purely infinite. In the first case, B is finite, so A is also finite (Proposition 6.15 of [8]). In the second case, every purely infinite simple unital C^* -algebra is tracially \mathcal{Z} -absorbing, as one sees by taking $\varphi = 0$ in Definition 2.1, and if B is purely infinite then so is A by Proposition 6.17 of [8].

Proof of Theorem 2.2. We verify the conditions in Definition 2.1. So let $\varepsilon > 0$, let $F \subset A$ be a finite set, let $x \in A_+ \setminus \{0\}$, and let $n \in \mathbb{Z}_{>0}$. Write $F = \{a_1, a_2, \dots, a_m\}$. Without loss of generality, we may assume that $\|a_j\| \leq 1$ for $j = 1, 2, \dots, m$. By Lemma 2.4 of [8], there exist $x_1, x_2 \in A_+ \setminus \{0\}$ such that

$$(2.1) \quad x_1 \sim x_2, \quad x_1 x_2 = 0, \quad \text{and} \quad x_1 + x_2 \in \overline{xAx}.$$

We may clearly assume that $\|x_1\| = \|x_2\| = 1$.

Apply Lemma 1.3 with $\frac{\varepsilon}{3}$ in place of ε , obtaining $\delta_0 > 0$. Set $\delta_1 = \min(\frac{\varepsilon}{3}, \frac{\delta_0}{3}) > 0$. Apply Lemma 1.2 with δ_1 in place of ε , with a_1, a_2, \dots, a_m as given, with $x_1 \in A_+ \setminus \{0\}$ in place of x , with $1_A \in B_+ \setminus \{0\}$ in place of y , and with $N = 1$, to obtain $c_1, c_2, \dots, c_m \in A$ and $g_0, g_1 \in B$ such that:

- (1) $0 \leq g_1 \leq g_0 \leq 1$ and $g_0 g_1 = g_1$.
- (2) $\|c_j\| \leq \|a_j\|$ for $j = 1, 2, \dots, m$.
- (3) $\|c_j - a_j\| < \delta_1$ for $j = 1, 2, \dots, m$.
- (4) $(1 - g_0)c_j, (1 - g_1)c_j \in B$ for $j = 1, 2, \dots, m$.
- (5) $g_k \lesssim_B 1_A$ and $g_k \lesssim_A x_1$ for $k = 0, 1$.
- (6) $\|g_0 a_j - a_j g_0\| < \delta_1$ and $\|g_1 a_j - a_j g_1\| < \delta_1$ for $j = 1, 2, \dots, m$.

Apply Lemma 5.3 of [8] with $r = 1_A$ to obtain $b \in B_+ \setminus \{0\}$ satisfying

$$(2.2) \quad b \lesssim_A x_2.$$

(We do not use condition (3) of that lemma, and we only use condition (1) of that lemma to guarantee that $b \neq 0$.) Apply Lemma 1.8 with n as given and with δ_1 in place of ε ; let $\delta_2 > 0$ be the resulting strictly positive number. Set $\delta_3 = \min(\delta_1, \delta_2) > 0$ and

$$S = \{g_0, 1 - g_0, (1 - g_0)^{1/2}\} \cup \{(1 - g_1)c_j : j = 1, 2, \dots, m\} \subset B.$$

Since B is tracially \mathcal{Z} -absorbing, there is a completely positive contractive order zero map $\varphi_0 : M_n \rightarrow B$ such that:

- (7) $1_A - \varphi_0(1) \lesssim_B b$.
- (8) $\|[\varphi_0(z), y]\| < \delta_3$ for all $z \in M_n$ with $\|z\| \leq 1$ and all $y \in S$.

For $z \in M_n$ with $\|z\| \leq 1$, we have $\|[\varphi_0(z), 1 - g_0]\| < \delta_3 \leq \delta_2$. We apply the choice of δ_2 using Lemma 1.8 with $1 - g_0$ in place of x and taking A and B there to be both equal to B . We get a completely positive contractive order zero map $\varphi : M_n \rightarrow B$ (which we regard as a map to A) such that $\|\varphi(z) - \varphi_0(z)(1 - g_0)\| < \delta_1$ for all $z \in M_n$ with $\|z\| \leq 1$ and such that

$$(2.3) \quad 1 - \varphi(1) \lesssim_B g_0 \oplus [1 - \varphi_0(1)].$$

For any such z , we compute, using (8) and the definition of S at the third step,

$$\begin{aligned} & \|\varphi(z) - (1 - g_0)^{1/2} \varphi_0(z) (1 - g_0)^{1/2}\| \\ & \leq \|\varphi(z) - \varphi_0(z)(1 - g_0)\| + \|\varphi_0(z)(1 - g_0) - (1 - g_0)^{1/2} \varphi_0(z) (1 - g_0)^{1/2}\| \\ & < \delta_1 + \|[\varphi_0(z), (1 - g_0)^{1/2}]\| \|(1 - g_0)^{1/2}\| \\ & < \delta_1 + \delta_3 \leq 2\delta_1 \leq \delta_0. \end{aligned}$$

For $j = 1, 2, \dots, m$, we want to apply the choice of δ_0 using Lemma 1.3 with c_j in place of c , with $\varphi(z)$ in place of z , and with $\varphi_0(z)$ in place of z_0 . We have just verified condition (7) of Lemma 1.3. Conditions (1), (2), (3), (5), and (6) of Lemma 1.3 follow from the requirements $\|a_j\| \leq 1$, $\|z\| \leq 1$, $\delta_1 \leq \delta_0$, the choice of S , and (1), (2), and (8) above. It remains to verify condition (4) of Lemma 1.3. Using (1), (3), and (6) above, for $k = 0, 1$ we get

$$\|[c_j, g_k]\| \leq 2\|c_j - a_j\|\|g_k\| + \|[a_j, g_k]\| < 2\delta_1 + \delta_1 \leq \delta_0,$$

as desired. The choice of δ_0 using Lemma 1.3 therefore implies $\|[\varphi(z), c_j]\| < \frac{\varepsilon}{3}$. We now estimate

$$\|[\varphi(z), a_j]\| \leq 2\|a_j - c_j\|\|\varphi(z)\| + \|[\varphi(z), c_j]\| < 2\delta_1 + \frac{\varepsilon}{3} \leq \varepsilon.$$

We have verified condition (2) in Definition 2.1.

It remains to show that $1_A - \varphi(1) \precsim_A x$. Using (2.3) at the first step, using (5) and (7) at the second step, using (2.2) at the third step, and using (2.1) at the fourth step, we get

$$1_A - \varphi(1) \precsim_B g_0 \oplus [1 - \varphi_0(1)] \precsim_A x_1 \oplus b \precsim_A x_1 \oplus x_2 \precsim_A x,$$

as required. \square

Theorem 2.3. Let A be a simple separable infinite dimensional nuclear unital C^* -algebra, and let B be a centrally large subalgebra of A . If $\mathcal{Z} \otimes B \cong B$ then $\mathcal{Z} \otimes A \cong A$.

Proof. Proposition 2.2 of [5] implies that B is tracially \mathcal{Z} -absorbing. So A is tracially \mathcal{Z} -absorbing by Theorem 2.2. Since A is nuclear, Theorem 4.1 of [5] implies that $\mathcal{Z} \otimes A \cong A$. \square

Theorem 2.4. Let A be a stably finite simple infinite dimensional unital C^* -algebra, and let B be a centrally large subalgebra of A . If A is tracially \mathcal{Z} -absorbing, then B is tracially \mathcal{Z} -absorbing.

As will be clear from the proof, we can drop the stable finiteness requirement if we require that B be stably centrally large in A .

Proof of Theorem 2.4. We verify the conditions in Definition 2.1. So let $\varepsilon > 0$, let $F \subset B$ be a finite set, let $x \in B_+ \setminus \{0\}$, and let $n \in \mathbb{Z}_{>0}$. Without loss of generality, we may assume that $\|a\| \leq 1$ for all $a \in F$. By Lemma 2.4 of [8], there exist $x_1, x_2, x_3 \in B_+ \setminus \{0\}$ such that

$$(2.4) \quad x_1 \sim x_2 \sim x_3, \quad x_1 x_2 = x_1 x_3 = x_2 x_3 = 0, \quad \text{and} \quad x_1 + x_2 + x_3 \in \overline{x B x}.$$

Apply Lemma 1.8 with n as given and with $\frac{\varepsilon}{4}$ in place of ε , getting $\delta_0 > 0$. Set $\delta = \min(\frac{\varepsilon}{4}, \delta_0)$.

Since A is tracially \mathcal{Z} -absorbing, there is a completely positive contractive order zero map $\varphi_0: M_n \rightarrow A$ such that:

- (1) $1 - \varphi_0(1) \precsim_A x_1$.
- (2) For $z \in M_n$ with $\|z\| \leq 1$ and $a \in F$, we have $\|\varphi_0(z)a - a\varphi_0(z)\| < \delta$.

Since B is centrally large in A , there exist $y_{j,k} \in A$ for $j, k = 1, 2, \dots, n$ and $g \in B$ such that:

- (3) $0 \leq g \leq 1$.
- (4) $\|y_{j,k} - \varphi_0(e_{j,k})\| < \delta$ for $j, k = 1, 2, \dots, n$.
- (5) $(1 - g)y_{j,k}^* \in B$ for $j, k = 1, 2, \dots, n$.
- (6) $g \precsim_B x_3$ and $g \precsim_A 1$.
- (7) $\|g\varphi_0(e_{j,k}) - \varphi_0(e_{j,k})g\| < \delta$ for $j, k = 1, 2, \dots, n$.
- (8) $\|ga - ag\| < \delta$ for $a \in F$.

By (5), for $j, k = 1, 2, \dots, n$ we have $y_{j,k}(1 - g) \in B$, so $\text{dist}(\varphi_0(e_{j,k})(1 - g), B) < \delta$. Since $\delta \leq \delta_0$, we can apply the choice of δ_0 using Lemma 1.8, taking A , B , and φ_0 there to be as given and with $x = 1 - g$. We get a completely positive contractive order zero map $\varphi: M_n \rightarrow B$ such that:

- (9) $\|\varphi(z) - \varphi_0(z)(1 - g)\| < \frac{\varepsilon}{4}$ for all $z \in M_n$ with $\|z\| \leq 1$.
- (10) $1 - \varphi(1) \precsim_A g \oplus [1 - \varphi_0(1)]$.

From (10), (1), and (6), we get $1 - \varphi(1) \precsim_A x_3 \oplus x_1 \sim_A x_1 + x_3$. Corollary 3.8 of [2] implies that B is stably centrally large in A . In particular, B is stably large in A . Using (2.4), apply Lemma 6.5 of [8] with $1 - \varphi(1)$ in place of a , with x in place of b , with $x_1 + x_3$ in place of c , and with x_2 in place of x , to get $1 - \varphi(1) \precsim_B x$. This is part (1) of Definition 2.1.

For part (2) of Definition 2.1, let $a \in F$ and let $z \in M_n$ satisfy $\|z\| \leq 1$. Then, using (9) at the second step, (8) and (2) at the third step, and $\delta \leq \frac{\varepsilon}{4}$ at the fourth step,

$$\begin{aligned} \|[\varphi(z), a]\| &\leq 2 \|\varphi(z) - \varphi_0(z)(1 - g)\| + \|[\varphi_0(z)(1 - g), a]\| \\ &< \frac{\varepsilon}{2} + \|\varphi_0(z)\| \| [1 - g, a] \| + \|[\varphi_0(z), a]\| \|1 - g\| \\ &< \frac{\varepsilon}{2} + \delta + \delta \leq \varepsilon. \end{aligned}$$

This completes the proof. \square

Corollary 2.5. Let A be a simple separable infinite dimensional unital C^* -algebra, and let B be a centrally large nuclear subalgebra of A . If $\mathcal{Z} \otimes A \cong A$ then $\mathcal{Z} \otimes B \cong B$.

Proof. The proof is essentially the same as that of Theorem 2.3, using Theorem 2.4 in place of Theorem 2.2. \square

Wei Sun ([15]) has studied a class of “generalized higher dimensional non-commutative tori”. They are defined in terms of generators and relations, involving n commuting unitaries (which generate a copy of $C((S^1)^n)$), a sequence $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n$, a further (nonunitary) element b which satisfies commutation relations with the given unitaries involving θ , and a nonnegative function $\gamma \in C((S^1)^n)$ related to the extent to which b fails to be unitary. The resulting algebra is called $A_{\theta, \gamma}$. Let $h: (S^1)^n \rightarrow (S^1)^n$ be the homeomorphism given by

$$h(z_1, z_2, \dots, z_n) = (e^{2\pi i \theta_1} z_1, e^{2\pi i \theta_2} z_2, \dots, e^{2\pi i \theta_n} z_n)$$

for $z_1, z_2, \dots, z_n \in S^1$. When $\theta_1, \theta_2, \dots, \theta_n$ are rationally independent, so that h is minimal, and when the zero set Y of γ satisfies $h^n(Y) \cap Y = \emptyset$ for all $n \in \mathbb{Z} \setminus \{0\}$, Wei Sun has shown that $A_{\theta, \gamma}$ is isomorphic to the subalgebra $C^*(\mathbb{Z}, (S^1)^n, h)_Y \subset C^*(\mathbb{Z}, (S^1)^n, h)$ as in Definition 7.3 of [8]. Accordingly, it is useful to obtain information about $C^*(\mathbb{Z}, (S^1)^n, h)_Y$ from knowledge of the structure of $C^*(\mathbb{Z}, (S^1)^n, h)$. The algebra $C^*(\mathbb{Z}, (S^1)^n, h)$ is a special example of a higher dimensional noncommutative torus.

Example 2.6. Let the notation be as in the preceding discussion, including the assumptions that $\theta_1, \theta_2, \dots, \theta_n$ are rationally independent and that $Y \subset (S^1)^n$ is a closed subset satisfying $h^n(Y) \cap Y = \emptyset$ for all $n \in \mathbb{Z} \setminus \{0\}$. Abbreviate $C^*(\mathbb{Z}, (S^1)^n, h)$ to A and $C^*(\mathbb{Z}, (S^1)^n, h)_Y$ to A_Y . It is known that A is \mathcal{Z} -stable (for example, see Corollary 3.4 of [16]), that A has a unique tracial state τ (Lemma 3.2(i) of [14]), and that $\tau_*(K_0(A))$ is dense in \mathbb{R} (this is true of its subalgebra A_{θ_1} , for which the range of the trace on K -theory is $\mathbb{Z} + \theta_1 \mathbb{Z}$ by Proposition 1.4 of [11]). Theorem 7.10 of [8] and Theorem 4.6 of [2] imply that A_Y is a centrally large subalgebra of A . So Corollary 2.5 implies that A_Y is \mathcal{Z} -stable. It now follows from Theorem 6.7 of [12] that A_Y has stable rank one. Theorem 6.2 of [8] implies that A_Y has a unique tracial state, namely $\sigma = \tau|_{A_Y}$. Suppose further that $K^1(Y) = 0$. It then follows from Theorem 2.4 of [10] and the discussion after Example 2.6 of [10]

that the map from $K_0(A_Y)$ to $K_0(A)$ is surjective. So $\sigma_*(K_0(A)_Y)$ is dense in \mathbb{R} . Now Corollary 7.3 of [12] implies that, in this case, A_Y has real rank zero. In particular, under these hypotheses, the algebra $A_{\theta, \gamma}$ of [15] is \mathcal{Z} -stable, has stable rank one, and, if $K^1(Y) = 0$, has real rank zero.

The hypothesis $K^1(Y) = 0$ is stronger than needed for the conclusion that A_Y has real rank zero. Indeed, under our other hypotheses, this may always be true. We leave such questions to [15].

There are other ways to get these results. For example, $C^*(\mathbb{Z}, (S^1)^n, h)_Y$ is a simple direct limit, with no dimension growth, of recursive subhomogeneous C^* -algebras.

In Theorem 2.3, it is not possible to replace \mathcal{Z} with a general strongly selfabsorbing C^* -algebra D .

Example 2.7. Let D be the 2^∞ UHF algebra. Let X be the Cantor set, and let $h: X \rightarrow X$ be the 2-odometer. (See page 332 of [9], or Section VIII.4 of [3].) Set $A = C^*(\mathbb{Z}, X, h)$, fix $y \in X$, and take $B = C^*(\mathbb{Z}, X, h)_{\{y\}}$, as in Definition 7.3 of [8]. (This is the algebra called $A_{\{h(y)\}}$ in Theorem 3.3 of [9]. One uses $h(y)$ rather than y because of the difference between the conventions used in [9] and [8].) Then $K_0(A) \cong \mathbb{Z}[\frac{1}{2}]$, as explained on page 332 of [9]. The inclusion of B in A is an isomorphism on K_0 , by Theorem 4.1 of [9], and B is an AF algebra by Theorem 3.3 of [9], so B is the 2^∞ UHF algebra. (This specific case is Example VIII.6.3 in [3].) In particular, B is D -absorbing. However, the Pimsner-Voiculescu exact sequence (see Theorem VIII.5.1 of [3]) easily shows that $K_1(A) \cong \mathbb{Z}$. The Künneth Theorem (Theorem 4.1 of [13]) therefore implies that A is not D -absorbing. The subalgebra B is centrally large in A by Theorem 7.10 of [8] and Theorem 4.6 of [2].

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